Sec 4.1: Review of Basic Calculus of Matrix Functions

Definition: A **Matrix Function** is a matrix whose entries are functions. In this class we will consider matrices whose entries are real valued functions of a real number t.

Ex1. Consider the matrix

$$M(t) = \begin{bmatrix} t^2 - t & 3 \\ t - 1 & t - 2 \end{bmatrix}$$

(a) Compute $M'(t)$ and $\int M(t)\ dt$.

(b) Compute $\int_0^1 M(t)\ dt$.

(c) For what values of t, $M(t)$ has inverse?

(d) Find $(M(t))^{-1}$, whenever it makes sense.

Note: In general we will use Gaussian elimination to compute $(M(t))^{-1}$. However, if M is a 2×2 matrix we have that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

BASIC RULES:

$$\frac{d}{dt}\Big\{A(t) \pm B(t)\Big\} = \frac{d}{dt}\Big\{A(t)\Big\} \pm \frac{d}{dt}\Big\{B(t)\Big\}$$

$$\frac{d}{dt}\Big\{A(t) \cdot B(t)\Big\} = A(t) \cdot B'(t) + A'(t) \cdot B(t)$$

$$\int \Big\{A(t) \pm B(t)\Big\}\ dt = \int A(t)\ dt \pm \int B(t)\ dt$$

$$\int_a^b \Big\{A(t) \pm B(t)\Big\}\ dt = \int_a^b A(t)\ dt \pm \int_a^b B(t)\ dt$$

WARNING: In general, the product of matrices is not commutative. That is, if P and Q are matrices, then PQ may not be QP. Hence, $\frac{d}{dt}\Big\{B(t) \cdot A(t)\Big\}$ may not be $A(t) \cdot B'(t) + A'(t) \cdot B(t)$.

Sec 4.2: First Order Linear System

Has the standard form:

$$\vec{Y}\,' = P(t) \cdot \vec{Y} + \vec{G}(t), \qquad a < t < b$$

where

$$\vec{Y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ . \\ . \\ y_n(t) \end{bmatrix}, \qquad P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ p_{31}(t) & p_{32}(t) & \cdots & p_{3n}(t) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \qquad \vec{G}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \\ . \\ . \\ g_n(t) \end{bmatrix}$$

Ex1. Write the following system as a first order linear system. Assume $0 < t < \infty$.

$$y_1' = \sin(t) \cdot y_1 + \frac{t}{t^2 - 2t + 8} \cdot y_2 + \ln(t)$$

$$y_2' = (2t+1) \cdot y_1 + \mathbf{e}^{-2t} \cdot y_2 + \cos(t)$$

Ex2. Write the following system as a first order linear system. Assume $-2 < t < 2$.

$$(t+2)y_1' = 3ty_1 + 5y_2$$

$$(t-2)y_2' = 2y_1 + 4ty_2$$

Ex3. Rewrite the scalar differential equation as a first order linear system:

$$y^{(3)} - t^2 y'' + 3ty' + 5y = \mathbf{e}^{-4t}$$

Sol.
Note that this is a third order **scalar** differential equation. Define a column vector $\vec{Y}(t)$ as follows:

$$\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$$

Differentiate the column vector $\vec{Y}(t)$ with respect to t:

$$\vec{Y}\,'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$$

Now use the definition of $\vec{Y}(t)$ and the fact that $y^{(3)} = t^2 y'' - 3ty' - 5y + \mathbf{e}^{-4t}$.

Definition: The **trace of a matrix** A denoted by $tr[A]$, is defined to be the sum of the elements of the diagonal of A.

Example: Consider A from our previous example,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $tr[A] = 0 + 5 + 1 = 6$.

Remember, given a 3×3 matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$,

determinant of B= det (B) $= a\times$ det $\begin{bmatrix} e & f \\ h & k \end{bmatrix} - b\times$ det $\begin{bmatrix} d & f \\ g & k \end{bmatrix} + c\times$det $\begin{bmatrix} d & e \\ g & h \end{bmatrix}$

Example: Calculate det (A).

Det (A)= $0\times$ det $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - 1\times$ det $\begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} + 0\times$ det $\begin{bmatrix} -6 & 5 \\ 0 & 0 \end{bmatrix} = 6$

Sec 4.3: First Order Homogeneous System

In this case $\vec{G}(t) = \vec{0}$, i.e.

$$\vec{Y}\,' = P(t) \cdot \vec{Y}, \quad a < t < b.$$

$\vec{y}\,' = P(t)\vec{y} + \vec{G}(t)$

consider homog

$\vec{G}(t) = \vec{0}$

1st order linear system

Motivation: Consider the I.V.P. $\vec{Y}' = A\vec{Y}$, $\vec{Y}(0) = \vec{Y}_0$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{Y}_0 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

- Show that

$$\vec{Y_1}(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 0 \end{bmatrix};\ \vec{Y_2}(t) = \begin{bmatrix} e^{3t} \\ 3e^{3t} \\ 0 \end{bmatrix};\ \vec{Y_3}(t) = \begin{bmatrix} 0 \\ 0 \\ e^{t} \end{bmatrix} \text{ are all solutions of the D.E. } \vec{Y}' = A\vec{Y}.$$

① $\vec{y}_1$ is a solution $\Leftrightarrow$ $\vec{y}_1' = P(t)\,\vec{y}_1(t) \Leftrightarrow \vec{y}_1' = A\vec{y}_1$

$$\vec{y}_1' = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 0 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0\cdot e^{2t} + 1\cdot 2e^{2t} + 0\cdot 0 \\ -6e^{2t} + 10e^{2t} + 0 \\ 0e^{2t} + 0\cdot 2e^{2t} + 1\cdot 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 0 \end{pmatrix}$$

$\Rightarrow \vec{y}_1$ is a solution

Is y_2 a solution?

$$\vec{y}_2' = \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \\ 0 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -6e^{3t} + 15e^{3t} \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 9e^{3t} \\ 0 \end{pmatrix} \Rightarrow \vec{y}_2 \text{ is a solution}$$

Is y_3 a solution?

$$\vec{y}_3' = \begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix} \Rightarrow \vec{y}_3(t) \text{ is a solution}$$

- The set $\{\vec{Y_1}(t), \vec{Y_2}(t), \vec{Y_3}(t)\}$ is a Fundamental Set of Solutions if every solution can be written as a linear combination of the functions in the set. Show that this set form a Fundamental Set for the IVP above.

(F.S.) Fundamental Set of solutions is a set of solutions $(\vec{y}_1, \vec{y}_2, \vec{y}_3)$ such that any solution of $\vec{y}\,' = A\vec{y}$ is a linear combination of the functions in the set.

Fundamental Set of Solutions → ① All the functions in the set are solutions
→ ② they are linearly independent

Def A general solution to $\vec{y}\,' = P(t)\,\vec{y}$ is a linear combination of the functions in a F.S.

$$\vec{y}\,'(t) = C_1\vec{y}_1(t) + C_2\vec{y}_2(t) + C_3\vec{y}_3(t)$$

The functions $\{\vec{Y}_1, \vec{Y}_2, \vec{Y}_3\}$ are linearly independent $\Leftrightarrow W(t) = \det[\vec{Y}_1, \vec{Y}_2, \vec{Y}_3] \neq 0$

- Solve the initial value problem.

(2) $\vec{Y}_1, \vec{Y}_2, \vec{Y}_3$ are LiId $\Leftrightarrow W(t) = \det\begin{bmatrix} e^{-2t} & e^{3t} & 0 \\ 2e^{2t} & 3e^{3t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \neq 0$ for any $t \in \mathbb{R}$ $\Rightarrow \{Y_1, Y_2, Y_3\}$ form a F.S.

$\det\begin{bmatrix} e^{-2t} & e^{3t} & 0 \\ 2e^{2t} & 3e^{3t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} = e^{t}\det\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} = e^{t}\left[3e^{5t} - 2e^{5t}\right] = e^{t}e^{5t} = e^{6t} \neq 0$ for any $t \in \mathbb{R}$

the general solution to $\vec{Y}' = AY$ is $Y(t) = C_1\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix} + C_2\begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix}$

How do I solve the IVP? $\vec{Y}(0) = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = C_1\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ 2C_1 + 3C_2 \\ C_3 \end{pmatrix}$

$3 = C_1 + C_2$
$7 = 2C_1 + 3C_2$
$4 = C_3$
$\Rightarrow C_1 = 2,\ C_2 = 1,\ C_3 = 4$

Fundamental Matrix. Given a homogeneous system:

$$\vec{Y}\,' = P(t) \cdot \vec{Y}, \quad a < t < b,$$

a matrix $\Phi(t)$ is called a fundamental matrix if:
(i) Each column of $\Phi(t)$ solves the homogeneous equation.
(ii) $\det(\Phi(t)) \neq 0$ for all t in (a, b)

Theorem. If $\Phi(t)$ is a fundamental matrix for the homogeneous equation $\vec{Y}\,' = P(t) \cdot \vec{Y}$, then the general solution is a linear combination of the columns of $\Phi(t)$. More precisely, the general solution is given by

$$\vec{Y} = \Phi(t) \cdot \vec{\mathbf{c}},$$

where $\vec{\mathbf{c}}$ is an arbitrary constant column vector $\vec{\mathbf{c}} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}^t$.

Ex1.
(a) Verify that the matrix $\Phi(t) = \begin{bmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{bmatrix}$ is a fundamental matrix for the homogeneous system

$$\vec{Y}\,' = \begin{bmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{bmatrix} \vec{Y}, \qquad 0 < t < \infty$$

$\vec{y}\,' = P(t)\vec{y}$

(1) Need to show that columns of $\Phi(t)$ are both solutions and that they are LI

$\Phi_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ is a solution $\Leftrightarrow \Phi_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{bmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{t} - \frac{1}{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $t \neq 0$

$\Phi_2(t) = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$ is a sol $\Leftrightarrow \Phi_2' = \begin{pmatrix} -t^{-2} \\ 2t^{-3} \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{bmatrix}\begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix} = \begin{pmatrix} -t^{-2} \\ t^{-3} + t^{-3} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -t^{-2} \\ 2t^{-3} \end{pmatrix}$ ✓ $\Phi_2(t)$ is a sol?

(2) $\det(\Phi(t)) \neq 0 \Leftrightarrow \det\begin{bmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{bmatrix} = -\frac{t}{t^2} - \frac{1}{t} = \frac{-2}{t} \neq 0$ for any $t \neq 0$

(b) Solve the initial value problem: $\vec{Y}\,' = \begin{bmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{bmatrix} \vec{Y}, \ \ \vec{Y}(1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

general solution for $\vec{y}' = P(t)\,y(t)$ is $\vec{Y}(t) = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ (general solution in matrix form)

$\vec{y}(t) = \Phi_1(t)\,c_1 + \Phi_2(t)\,c_2$

to solve IVP,

$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \vec{y}(1) = c_1\Phi_1(1) + c_2\Phi_2(1) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{cases} 2 = c_1 + c_2 \\ 3 = c_1 + c_2 \end{cases} \iff$ In matrix form $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1}A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/2 + 3/2 \\ 2/2 - 3/2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix}$

Theorem [Abel's Formula] If the column vectors $\vec{\psi}_1, \ \vec{\psi}_2, \ \cdots, \ \vec{\psi}_n$ are solutions of the homogeneous system

$$\vec{Y}\,' = P(t)\cdot\vec{Y}, \qquad a < t < b,$$

then the determinant of the matrix $\Psi(t) = \begin{bmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \cdots & \vec{\psi}_n \end{bmatrix}$ is given by

$$\det(\Psi(t)) = \det(\Psi(t_0))\exp\Big(\int_{t_0}^{t} \mathrm{tr}\{P(s)\}\,ds\Big)$$

where t_0 is arbitrary in the interval (a, b). The determinant of the matrix Ψ(t) is called the Wronskian of the system.

Using Abel's Formula $\det(\Phi(t)) \neq 0 \iff \det(\Phi(t_0)) \neq 0$

$\det(\Phi(1)) = \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \neq 0 \iff \det(\Phi(t)) \neq 0 \iff \Phi$ is a FMatrix ← ? DMF

Ex2. Suppose $\Phi(t)$ is a fundamental matrix for the homogeneous system

$$\vec{Y}\,' = \begin{bmatrix} \dfrac{\cos(t^3)}{t^2+1} & 1 \\ \sin(t) & \dfrac{2t-\cos(t^3)}{t^2+1} \end{bmatrix} \vec{Y} \iff \vec{Y}\,' = P(t)\vec{Y}$$

If $\Phi(0)$ is the identity matrix, what is $\det(\Phi(t))$? what is $\det(\Phi(7))$?

$$\int_0^t \frac{2s}{s^2+1}\,ds = \int_1^{t^2+1} \frac{1}{u}\,du = \ln(u)\Big|_1^{t^2+1} \Rightarrow \ln(t^2+1) - \ln(1)$$

$u = s^2+1$, $du = 2s\,ds$, $\ln(1) = 0$

$$\det(\Phi(t)) = \det I \, e^{\int_0^t \frac{2s}{s^2+1}\,ds} = e^{\ln(t^2+1)} = t^2+1 \neq 0$$

($I = \Phi(0)$)

$$\det(\Phi(7)) = 50$$

Corollary. If the columns of $\Psi(t) = \begin{bmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \cdots & \vec{\psi}_n \end{bmatrix}$ are solutions of the homogeneous system

$$\vec{Y}\,' = P(t)\cdot\vec{Y}, \qquad a < t < b$$

and $\det(\Psi(t_0)) \neq 0$ for some t_0 in (a, b), then $\Psi(t)$ is a fundamental matrix.

Proof:

Ex3. It is known that

$$\vec{\psi}_1 = \begin{bmatrix} \mathbf{e}^{-t} \\ 0 \\ \mathbf{e}^{-t} \end{bmatrix}, \quad \vec{\psi}_2 = \begin{bmatrix} -\mathbf{e}^{t}\sin(2t) \\ \mathbf{e}^{t}\cos(2t) \\ 2\mathbf{e}^{t}\cos(2t) \end{bmatrix} \quad \text{and} \quad \vec{\psi}_3 = \begin{bmatrix} \mathbf{e}^{t}\cos(2t) \\ \mathbf{e}^{t}\sin(2t) \\ 2\mathbf{e}^{t}\sin(2t) \end{bmatrix}$$

are solutions of the homogeneous system

$$\vec{Y}\,' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & -2 \\ 4 & 12 & -5 \end{bmatrix} \vec{Y}, \qquad -\infty < t < \infty.$$

Let $\Psi(t) = \begin{bmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \vec{\psi}_3 \end{bmatrix}$. What is $\det(\Psi(t))$? What can you say about the matrix $\Psi(t)$?